

A NEW SOLVABILITY CRITERION FOR FINITE GROUPS

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ABSTRACT. In 1968, John Thompson proved that a finite group G is solvable if and only if every 2-generator subgroup of G is solvable. In this paper, we prove that solvability of a finite group G is guaranteed by a seemingly weaker condition: G is solvable if, for all conjugacy classes C and D of G consisting of elements of prime power order, *there exist* $x \in C$ and $y \in D$ for which $\langle x, y \rangle$ is solvable. We also prove the following property of finite nonabelian simple groups, which is the key tool for our proof of the solvability criterion: if G is a finite nonabelian simple group, then there exist two prime divisors a and b of $|G|$ such that, for all elements $x, y \in G$ with $|x| = a$ and $|y| = b$, the subgroup $\langle x, y \rangle$ is not solvable. Further, using a recent result of Guralnick and Malle, we obtain a similar membership criterion for any family of finite groups closed under forming subgroups, quotients and extensions.

1. INTRODUCTION

John G. Thompson’s famous “N-group paper” [T] of 1968 included the following important solvability criterion for finite groups:

A finite group is solvable if and only if every pair of its elements generates a solvable group.

P. Flavell [F] gave a relatively simple proof of Thompson’s result in 1995. We prove that solvability of finite groups is guaranteed by a seemingly weaker condition than the solvability of all its 2-generator subgroups.

Theorem A. *Let G be a finite group. The following are equivalent:*

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- (1) G is solvable;
- (2) For all $x, y \in G$, there exists an element $g \in G$ for which $\langle x, y^g \rangle$ is solvable; and
- (3) For all $x, y \in G$ of prime power order, there exists an element $g \in G$ for which $\langle x, y^g \rangle$ is solvable.

Theorem A can be rephrased as the following essentially equivalent result.

Theorem A'. *Let G be a finite group such that, for all distinct conjugacy classes C and D of G consisting of elements of prime power order, there exist $x \in C$ and $y \in D$ for which $\langle x, y \rangle$ is solvable. Then G is solvable.*

Our second main result, which is the key tool for proving Theorem A, deals with the nonsolvability of certain 2-generator subgroups of finite nonabelian simple groups. Using the classification of finite simple groups, we prove the following theorem.

Theorem B. *Let G be a finite nonabelian simple group. Then there exist distinct prime divisors a, b of $|G|$ such that, for all $x, y \in G$ with $|x| = a$, $|y| = b$, the subgroup $\langle x, y \rangle$ is nonsolvable.*

In an earlier version of this paper by three of the authors [DHP], Theorem B was proved with the assumption of a and b prime replaced by the assumption that they be orders of elements of G (and the result with primes was conjectured). This weaker version of Theorem B led in [DHP] to a proof of the equivalence of conditions (1) and (2) of Theorem A, and to a proof of Theorem A' where C and D are arbitrary conjugacy classes.

Various results about nonabelian simple groups produce generating element pairs. However, in Theorem B we cannot in general choose primes a, b such that the nonsolvable subgroups $\langle x, y \rangle$ are all equal to G : for example, for the alternating group A_n , where $n > 5$ and n is not prime, it is easy to see that for any primes a, b less than n , there exist $x, y \in A_n$ with $|x| = a$, $|y| = b$ and $\langle x, y \rangle \neq A_n$. On the other hand, for many of the finite simple groups G of Lie type, we can choose primes a and b such that $G = \langle x, y \rangle$ for any $x, y \in G$ of orders a and b respectively. We discuss this property in the final section and show that there are also infinitely many finite simple groups of Lie type for which no choice of primes a, b gives the stronger “generation result”.

Using a recent result of Guralnick and Malle [GM2, Theorem 1.2] together with the methods used to derive Theorem A from Theorem B, we can prove a stronger version of Theorem A.

Theorem C. *Let \mathcal{X} be a family of finite groups which is closed under taking subgroups and quotient groups, and forming extensions. Then a finite group G is in \mathcal{X} if and only if, for every pair of conjugacy classes C and D of G , there exist $x \in C$ and $y \in D$ for which $\langle x, y \rangle \in \mathcal{X}$.*

Using standard reduction techniques, we have the following easy corollary of Theorem A for *linear groups*, that is, subgroups of $\mathrm{GL}(n, K)$ for some n and field K .

Corollary D. *Let G be a finitely generated linear group. Then G is solvable if and only if, for all $x, y \in G$, there exists $g \in G$ such that $\langle x, y^g \rangle$ is solvable.*

Note that the finite generation hypothesis cannot be removed. Suppose that G is a simple algebraic group (and so in particular a linear group). If $x, y \in G$, then there exists $g \in G$ with $\langle x, y^g \rangle$ contained in a Borel subgroup (and so solvable). Indeed, if we take G be a simple compact Lie group, then any element is contained in a maximal torus and so given $x, y \in G$, x and y^g will commute for some $g \in G$.

Theorem A can also be used to give in Corollary E a characterization of finite nilpotent groups, and our proof depends on the finite simple group classification, since Theorem A does. It would be interesting to see if Corollary E could be proved without the classification of finite simple groups. One can also deduce Corollary E from the result of Fein, Kantor and Schacher [FKS] that in any transitive action of a finite group, there exists a fixed point free element of prime power order. (However this theorem in [FKS] is actually more difficult to prove than Theorem A.)

Corollary E. *Let G be a finite group. Then G is nilpotent if and only if for every pair of distinct primes p and q and for every pair of elements $x, y \in G$ with x a p -element and y a q -element, x and y^g commute for some $g \in G$.*

We can restate Theorem A in an analogous manner:

Corollary F. *Let G be a finite group. Then G is solvable if and only if for every pair of distinct primes p and q and for every pair of elements $x, y \in G$ with x a p -element and y a q -element, $\langle x, y^g \rangle$ is a $\{p, q\}$ -group for some $g \in G$.*

We discuss various other generalizations of Thompson's theorem in the next section. We prove Theorem B for alternating groups and sporadic groups in Section 3 and for the groups of Lie type in Section 4. We conclude the latter section with the proof of Theorem B. In Section

5, we deduce Theorems A and C and Corollaries D, E and F. In the final section, we give some examples and remarks.

We note that all the main results depend upon the classification of finite simple groups. However, the proofs of Theorem B for the known simple groups do not use the classification. This is in contrast to Theorem C where we need detailed information about the maximal subgroups of the finite groups of Lie type (which also depends upon the classification).

2. OTHER GENERALISATIONS OF THOMPSON'S THEOREM

Several other “Thompson-like” results have appeared in the literature recently. We mention here four such theorems. In the first three results, solvability of all 2-generator subgroups is replaced by a weaker condition restricting the required set of solvable 2-generator subgroups, in different ways from our generalisation.

In 2000, Wilson and the second author [GW] obtained a solvability criterion by restricting the proportion of 2-generator subgroups required to be solvable.

Theorem 2.1. *A finite group is solvable if and only if more than $\frac{11}{30}$ of the pairs of elements of G generate a solvable subgroup.*

In addition they proved similar results showing that the properties of nilpotency and having odd order are also guaranteed if a sufficient proportion of element pairs generate subgroups with these properties, namely more than $\frac{1}{2}$ for nilpotency, and more than $\frac{11}{30}$ for having odd order.

In contrast to this, in a paper published in 2009, Gordeev, Grunewald, Kunyavskii and Plotkin [GGKP1] proved a solvability criterion which involved 2-generation within each conjugacy class. This result was also proved independently by Guest [G2, Corollary 1] (see also [G1] for related results).

Theorem 2.2. *A finite group G is solvable if and only if, for each conjugacy class C of G , each pair of elements of C generates a solvable subgroup.*

A stronger result of this type follows immediately from Guest [G1, Theorem 3], while a slightly weaker version was obtained recently by Kaplan and Levy in [KL, Theorem 4]. This criterion involves only a limited 2-generation within the conjugacy classes of elements of odd prime-power order.

Theorem 2.3. *A finite group G is solvable if and only if, for all $x, y \in G$ with x a p -element, for some prime $p > 3$, and y a 2-element, the group $\langle x, x^y \rangle$ is solvable.*

Our requirement, while ranging over all conjugacy classes, requires only *existence* of a solvable 2-generator subgroup with one generator from each of two classes. We know of no similar criteria in this respect.

The fourth result we draw attention to is in a 2006 paper of Kunyavskii, Plotkin, Shalev and the second author [GKPS]. They proved that membership of the solvable radical of a finite group is characterised by solvability of certain 2-generator subgroups. (The *solvable radical* $R(G)$ of a finite group G is the largest solvable normal subgroup of G .)

Theorem 2.4. *For a finite group G , the solvable radical $R(G)$ coincides with the set of all elements $x \in G$ with the property: “for any $y \in G$, the subgroup $\langle x, y \rangle$ is solvable”.*

Further results in this direction may be found in [GGKP2] and in [G3].

In view of Theorem A and Theorem 2.4, it might seem reasonable to conjecture that the solvable radical of a finite group G is the set of $x \in G$ such that for any $y \in G$ there exists $g \in G$ making the group $\langle x, y^g \rangle$ solvable. However, this conjecture is false. For example, the group A_5 contains solvable subgroups of order 6 and 10, so if $x \in A_5$ is of order 2, then it satisfies the above conjecture, while it certainly does not belong to the solvable radical of A_5 . The same holds for elements of order 3 in $PSL(2, 7)$ and Simon Guest and the fourth author have constructed such counterexamples for elements x of an arbitrary prime order.

3. ALTERNATING AND SPORADIC SIMPLE GROUPS

We first note the following two lemmas.

Lemma 3.1. *Let H be a finite solvable group with order divisible by distinct primes p and q . Then H contains a subgroup of exponent pq , and of order pq^a or $p^a q$, for some positive integer a . In particular, if the Sylow p - and q -subgroups of H are both cyclic, then H contains a subgroup of order pq .*

Proof. The group H contains a Hall $\{p, q\}$ -subgroup, so we may assume that $|H| = p^b q^c$ for some positive integers b and c . Let N be a minimal normal subgroup of H . Interchanging p and q if necessary, we may assume that N is elementary abelian of order p^a for some positive integer a . Thus H contains a subgroup K containing N and of order

$p^a q$. This subgroup K must have exponent pq , and if the Sylow p -subgroups of H are cyclic then $|N| = p^a = p$ and $|K| = pq$. \square

Lemma 3.2. *Let H be a finite group and let p, q be distinct prime divisors of $|H|$. Assume that*

- (1) *Sylow q -subgroups of H are cyclic and Sylow p -subgroups of H have order p^s ;*
- (2) *p does not divide $q - 1$;*
- (3) *for $1 \leq m \leq s$, q does not divide $p^m - 1$ (this certainly holds if $q > p^s$); and*
- (4) *H contains no elements of order pq .*

Then H contains no subgroup of order $p^a q^b$ with $a, b > 0$. In particular, H is not solvable.

Proof. Suppose, to the contrary, that H contains a subgroup B of order $p^a q^b$ with $a, b > 0$. Let N be a minimal normal subgroup of B . Since B is solvable, N is elementary abelian. If N is a q -group, then by (1) $|N| = q$ and B contains a subgroup $M > N$ of order pq . If M is nonabelian, then p divides $q - 1$, in contradiction to (2), while if M is abelian, then it is cyclic of order pq , in contradiction to (4). Thus N is not a q -group. If N is a p -group, then by (1) $|N| = p^i \leq p^s$ and hence B contains a subgroup $M > N$ of order qp^i . By (4), an element of order q in M acts fixed point freely on N , which implies that q divides $p^i - 1$, in contradiction to (3). Thus B does not exist, as required. \square

Theorem B for the alternating groups follows from the following proposition.

Proposition 3.3. *For all $n \geq 5$, there exist distinct primes p and q satisfying $n/2 \leq p < q \leq n$ such that, for all $x, y \in A_n$ with $|x| = p$ and $|y| = q$, the subgroup $\langle x, y \rangle \cong A_d$ for some $d \geq q$. In particular, $\langle x, y \rangle$ is nonsolvable.*

Proof. Note that if m is a positive integer and $\pi(m)$ denotes the number of primes at most m , then the following is known (see, for example [Tr, Theorem 32]):

$$\pi(2m) - \pi(m) > m/(3 \ln 2m) \text{ for } m > 1.$$

Now, $m/(3 \ln 2m)$ is at least 2 or 3 for $m \geq 9$ or 23 respectively. It follows, by checking small values of n , that $\pi(n) - \pi(n/2) \geq 2$ for $n \geq 5$, with the exception of $n = 6$ or 10. Thus for all $n \geq 5$, there are primes p, q with $\frac{n}{2} \leq p < q \leq n$. In each case we choose q to be the largest prime at most n , and if $n \neq 6, 10$ we choose p to be the smallest prime greater than $\frac{n}{2}$; while for $n = 6, 10$ we choose $p = 3, 5$ respectively.

Let $x, y \in A_n$ with $|x| = p$ and $|y| = q$. As $q > n/2$, the Sylow q -subgroup of A_n is cyclic and as $p + q > n$, A_n contains no elements of order pq . Moreover, either $p > n/2$, or $p = n/2$ implying that $q < n$. Thus $q - 1 > p > q/2$, whence p does not divide $q - 1$ and q does not divide $p^2 - 1$. Hence, by Lemma 3.2, $\langle x, y \rangle$ is nonsolvable.

To prove the stronger assertion, note that $\langle x, y \rangle$ has an orbit of length $d \geq q$ and fixes each point outside this orbit. Now $d \leq n$, and so $\frac{d}{2} \leq \frac{n}{2} \leq p < q \leq d$. Moreover if either $d < n$ or $n \neq 6, 10$, then we have $p > \frac{d}{2}$. If $\frac{d}{2} < p \leq d - 3$ then it follows from a theorem of Jordan (see [Wi, Theorem 13.9]) that $\langle x, y \rangle \cong A_d$. Clearly $\frac{d}{2} < p \leq d - 3$ holds if $n \geq 46$ and it turns out, by checking smaller values of n , that the cases not covered by this argument, for our choices of p, q , are $5 \leq n \leq 10$ and $15 \leq n \leq 16$. In these cases, ad hoc arguments with the primes 11, 13 for $15 \leq n \leq 16$, the primes 5, 7 for $7 \leq n \leq 10$, and the primes 3, 5 for $n = 5, 6$, show that either $\langle x, y \rangle \cong A_d$, or $n = d = 6$ and $\langle x, y \rangle \cong \text{PSL}_2(5) \cong A_5$. \square

For each sporadic group G , including ${}^2F_4(2)'$, we can choose, using the [ATLAS], two primes p and q such that Lemma 3.2 applies and hence if $x, y \in G$ are of order p and q , respectively, then $\langle x, y \rangle$ is nonsolvable. More precisely, we give below the list (G, p^a, q^b) for sporadic groups G , including ${}^2F_4(2)'$, where p and q are primes satisfying Lemma 3.2 and the corresponding Sylow subgroups of G have order p^a and q^b , respectively.

$$M_{11}, 3^2, 11; M_{12}, 3^3, 11; M_{22}, 7, 23; M_{23}, 7, 23; M_{24}, 7, 23; J_1, 11, 19.$$

$$J_2, 3^3, 7; J_3, 17, 19; J_4, 37, 43; HS, 7, 11; He, 3^3, 17; McL, 7, 11.$$

$$Suz, 11, 13; Ly, 37, 67; Ru, 13, 29; O'N, 19, 31; Co_1, 13, 23; Co_2, 7, 23.$$

$$Co_3, 7, 23; Fi_{22}, 11, 13; Fi_{23}, 17, 23; Fi'_{24}, 23, 29; HN, 11, 19; Th, 19, 31.$$

$$B, 31, 47; M, 59, 71; {}^2F_4(2)', 5^2, 13.$$

In fact, for each sporadic group we can choose primes p, q so that $\langle x, y \rangle$ is not only nonsolvable but also simple for x of order p and y of order q (see [DHP, Proposition 2.2]).

Thus, we have shown:

Proposition 3.4. *Theorem B holds for the alternating simple groups, for the sporadic simple groups and for ${}^2F_4(2)'$.*

4. GROUPS OF LIE TYPE

In the following, $q = p^k$ is a power of a prime p . For any positive integer e , we say that a prime r is a *primitive prime divisor* of $q^e - 1$ if r divides $q^e - 1$ and r does not divide $q^i - 1$ for any positive integer $i < e$. Observe that then e is the order of q modulo the prime r ; so e divides $r - 1$ and, in particular, $r \geq e + 1$. The set of primitive prime divisors of $q^e - 1$ will be denoted by $\text{ppd}(q, e)$.

The following result of Zsigmondy [Z] will be used frequently.

Theorem 4.1. *Let $q \geq 2$ and $e \geq 2$. There exists a primitive prime divisor of $q^e - 1$ unless*

- (i) $q = 2^a - 1$ is a Mersenne prime and $e = 2$; or
- (ii) $(q, e) = (2, 6)$.

Let \mathbb{G} be a simple algebraic group over an algebraically closed field of positive characteristic. If F is an endomorphism of \mathbb{G} with set of fixed points $G := \mathbb{G}^F$ finite, then G is said to be a finite group of Lie type. In essentially all cases, the derived group of G modulo its center is simple. These are called the finite simple groups of Lie type. We will prove Theorem B for these groups (and so, by the classification of finite simple groups and Propositions 3.4, complete the proof of Theorem B). Indeed, we will assume that \mathbb{G} is simply connected (for example, we will take $\mathbb{G} = \text{SL}_n$ rather than PGL_n). Then G will be a perfect group, and we will say also that G is simply connected. It suffices to prove the result in these simply connected cases.

We refer the reader to [Ca1] or [GLS3] for basic facts about these groups. Let \mathbb{T} be an F -stable maximal torus of \mathbb{G} . Then $T := \mathbb{T}^F = \mathbb{T} \cap G$ is called a maximal torus of G . The torus T is said to be nondegenerate if $\mathbb{T} = C_{\mathbb{G}}(T)$. Every semisimple element of G is contained in a maximal nondegenerate torus (and this will be obvious in the cases we require). The Weyl group W of \mathbb{G} is the normalizer of \mathbb{T} modulo its centralizer. If T is a nondegenerate maximal torus of G , then $N_G(T)/C_G(T)$ embeds in the Weyl group W (see [Ca2, Proposition 3.3.6]).

Lemma 4.2. *Let G be a simply connected finite simple group of Lie type. Let $x \in G$ be a semisimple element of prime order r . If $y \in G$ has prime order s with y normalizing $\langle x \rangle$, then either $xy = yx$ or s divides $|W|$.*

Proof. Suppose that $xy \neq yx$. Then $xyx^{-1} = x^e$ where e has multiplicative order s modulo r . Let \mathbb{T} be a maximal torus of the algebraic group \mathbb{G} such that $G = \mathbb{G}^F$ and $x \in \mathbb{T}$. Since x and x^e are conjugate,

it follows that x and x^e are conjugate in $N_{\mathbb{G}}(\mathbb{T})$ [GLS3, 4.1.3] (this is just the fact that all maximal tori in $C_{\mathbb{G}}(x)$ are conjugate in $C_{\mathbb{G}}(x)$). Thus, $x^e = uxu^{-1}$ for some $u \in N_{\mathbb{G}}(\mathbb{T})$. It follows that the order of $uC_{\mathbb{G}}(\mathbb{T}) \in N_{\mathbb{G}}(\mathbb{T})/C_{\mathbb{G}}(\mathbb{T}) \cong W$ is a multiple of s , whence s divides $|W|$. \square

4.1. Finite classical groups. We now consider the finite classical groups arising from the simply connected classical groups \mathbb{G} (see [GLS3, Theorem 1.10.7 and pp. 69, 71]), namely $\mathrm{SL}_n(q)$ ($n \geq 2$), $\mathrm{SU}_n(q)$ ($n \geq 3$), $\mathrm{Sp}_n(q)$, ($n = 2m \geq 4$), $\mathrm{Spin}_n(q)$ ($n \geq 7$, nq odd) or $\mathrm{Spin}_n^{\pm}(q)$ ($n \geq 8$ even). Let G be one of these groups and let V denote the natural module for G . So V is an n -dimensional vector space over \mathbb{F}_q (or over \mathbb{F}_{q^2} in the case of $\mathrm{SU}_n(q)$). In the case $G = \mathrm{Spin}_n(q)$, V will not be a faithful module.

The simple idea to prove Theorem B is as follows. We choose elements $x_1, x_2 \in G$ of prime orders r_1 and r_2 which leave invariant irreducible submodules of V of dimensions n_1, n_2 respectively, where $n/2 < n_1 < n_2 \leq n$. Moreover, if possible we take $n_2 - n_1 > 1$. This is equivalent to saying that $r_1 r_2$ is a divisor of $|G|$, and that each r_i is a primitive prime divisor of $q^{n_i} - 1$ (or of $q^{2n_i} - 1$ if $G = \mathrm{SU}_n(q)$). In particular $r_i \equiv 1 \pmod{n_i}$ so $r_i \geq n_i + 1$. By considering the formula for the orders of the groups and Zsigmondy's Theorem 4.1, this (with $n_2 - n_1 > 1$) can always be done unless one of the following holds:

- (1) $G = \mathrm{SL}_n(q)$ with $n = 2, 3$ or 4 ;
- (2) $G = \mathrm{SU}_n(q)$ with $n = 3, 4$ or 6 ;
- (3) $G = \mathrm{Sp}_4(q)$;
- (4) $G = \mathrm{Spin}_8^+(q)$;
- (5) $G = \mathrm{SL}_6(2)$;
- (6) $G = \mathrm{Sp}_6(2)$ or $\mathrm{Sp}_8(2)$; or
- (7) $G = \mathrm{Spin}_8^-(2)$;

Let us exclude these cases for the moment. Then for each i , the Sylow r_i -groups of G are cyclic (because a faithful module for any non-cyclic r_i -group would have dimension at least $2n_i$ which is greater than $\dim V$). By Lemma 3.1, it suffices to prove that G has no subgroup of order $r_1 r_2$. Suppose to the contrary that $X = \langle x_1, x_2 \rangle$ is a subgroup of order $r_1 r_2$ with $|x_i| = r_i$. Let $V_i = [x_i, V]$. Then x_i acts irreducibly on V_i . Now x_1 and x_2 do not commute, for if did then $V_1 \cap V_2$ would be non-zero and invariant under each x_i , contradicting the fact that x_i is irreducible on V_i . Set $X_i := \langle x_i \rangle$ and note that one of X_1 or X_2 is normal in X . Now $N_G(X_i)/C_G(X_i)$ is cyclic of order dividing n_i (or $2n_i$ in the case of $\mathrm{SU}_n(q)$ – to see this, work in $\mathrm{GL}(V)$). Since $r_2 > n_2 > n_1$, this implies that X_2 is normal in X and $r_1 | n_2$. However

$n_2 > n_1 > n/2$ and $r_1 \equiv 1 \pmod{n_1}$ imply that $n_2 < 2n_1 < 2r_1$, whence $n_2 = r_1 = n_1 + 1$, contradicting $n_2 - n_1 > 1$.

(*) Indeed, we note that the argument applies even for $n_2 = n_1 + 1$ unless $n_2 = r_1$ is prime.

Therefore, to complete the proof of Theorem B for classical groups we are left with cases (1)–(7) above. For these groups, the following argument will help. If H is a subgroup of G , we call $N_G(H)/C_G(H)$ the *automizer* of H in G . We recall that, if G is finite simple group of Lie type in characteristic p , a parabolic subgroup of G is any subgroup which contains the normalizer of a Sylow p -subgroup of G . A prime r does not divide the order of any proper parabolic subgroup of G if and only if r does not divide the order of the normalizer in G of any nontrivial p -subgroup of G (see [GLS3, Theorem 3.1.3]).

Lemma 4.3. *Let G be a finite simple group of Lie type in characteristic p . Let r be a prime such that a Sylow r -subgroup is cyclic and r does not divide the order of any parabolic subgroup of G . Assume also that p does not divide the order of the automizer in G of a cyclic group of order r . Then $\langle x, y \rangle$ is nonsolvable for any $x, y \in G$ with $|x| = p$ and $|y| = r$.*

Proof. Suppose this is not the case, and let H be a minimal solvable subgroup of G of order a multiple of pr . Then H is a $\{p, r\}$ -group. If $O_p(H) \neq 1$, then it follows from the Borel-Tits lemma [GLS3, 3.1.3] that H is contained in a parabolic subgroup, which is a contradiction. Hence $O_p(H) = 1$ and so $O_r(H) \neq 1$. Since a Sylow r -subgroup is cyclic, $O_r(H)$ is cyclic, and an element $x \in H$ of order p normalises the unique subgroup $\langle y \rangle$ of $O_r(H)$ of order r . By the minimality of H it follows that $H = \langle y \rangle \cdot \langle x \rangle$ of order pr . However, since p does not divide the order of the automizer of $\langle y \rangle$ in G , it follows that x centralises y and so $O_p(H) = \langle x \rangle$, which is a contradiction. \square

We note that (recalling Theorem 4.1) the previous lemma applies to prove Theorem B when:

- (1') $G = \mathrm{SL}_n(q)$ or $\mathrm{Sp}_n(q)$ as long as p does not divide n and $(n, q) \neq (2, 2^a - 1)$;
- (2') $G = \mathrm{SU}_{2m}(q)$ or $\mathrm{SU}_{2m-1}(q)$ as long as p does not divide $2m - 1$;
- (3') $G = \mathrm{Spin}_{2m+1}(q)$ or $\mathrm{Spin}_{2m}^-(q)$ as long as p does not divide $2m$;
- (4') $G = \mathrm{Spin}_{2m}^+(q)$ as long as p does not divide $2m - 2$;

taking r a primitive prime divisor of $q^n - 1$, $q^{2(2m-1)} - 1$, $q^{2m} - 1$, and $q^{2m-2} - 1$, respectively, since in these cases r does not divide the order

of any parabolic subgroup of G . Comparing this list with the cases (1)–(7), we have now proved Theorem B for classical groups except in the following cases:

$\mathrm{SL}_n(q)$	$(n, q) =$	$(2, 2^a), (2, 2^a - 1), (3, 3^a), (4, 2^a), (6, 2)$
$\mathrm{SU}_n(q)$	$(n, q) =$	$(3, 3^a), (4, 3^a), (6, 5^a)$
$\mathrm{Sp}_n(q)$	$(n, q) =$	$(4, 2^a), (6, 2), (8, 2)$
$\mathrm{Spin}_8^\varepsilon(q)$	$(\varepsilon, q) =$	$(+, 2^a), (+, 3^a), (-, 2)$

We now handle these special cases. For each group G we choose distinct prime divisors r_1, r_2 of $|G|$ and show that $\langle x_1, x_2 \rangle$ is nonsolvable whenever $x_1, x_2 \in G$ with $|x_1| = r_1, |x_2| = r_2$ (modulo $Z(G)$). Often the Sylow r_i -subgroups are cyclic and we show that G has no subgroups of order $r_1 r_2$ and apply Lemma 3.1.

$G = \mathrm{SL}_2(q)$ with $q = 2^a \geq 4$ or $q = 2^a - 1 \geq 7$.

If $q = 2^a$, we can choose odd primes r_1 dividing $q - 1$ and r_2 dividing $q + 1$. The centralizers of the x_i then have orders $q \pm 1$ and their automizers have order 2. Thus, G has no subgroups of order $r_1 r_2$, while the Sylow r_i -subgroups are cyclic.

If $q = 2^a - 1$ is a Mersenne prime, then we take $|x_1| = r_1 = q$ and x_2 of order 4 (so of order 2 in the simple group). The only maximal subgroup of G containing x_1 has order $q(q - 1)$ and so contains no element of order 4. Thus $\langle x_1, x_2 \rangle = G$, and in particular is nonsolvable.

$G = \mathrm{SL}_3(q)$ with $q = 3^a \geq 3$.

If $q > 3$ then, by Theorem 4.1, $q^3 - 1$ and $q^2 - 1$ have primitive prime divisors r_2, r_1 respectively, and as noted in (*) the general argument works, proving Theorem B for $q > 3$. If $q = 3$, we take $r_1 = 2$ and $r_2 = 13$. The only maximal subgroup of order divisible by 13 has order 39, and hence $\langle x_1, x_2 \rangle = G$.

$G = \mathrm{SL}_4(q)$ with $q = 2^a \geq 2$.

Here $q^3 - 1$ and $q^4 - 1$ have primitive prime divisors r_1, r_2 respectively, and by (*) the general argument works, since 4 is not prime.

$G = \mathrm{SL}_6(2)$.

Take $r_1 = 31$ and $r_2 = 7$. There are no subgroups of order $7^a \cdot 31$ for any $a \geq 1$, whence the result follows from Lemma 3.1.

$G = \mathrm{SU}_3(q)$ with $q = 3^a \geq 3$.

If $q > 3$, then $q^2 - 1$ and $q^6 - 1$ have primitive prime divisors r_1, r_2

respectively, and the Sylow r_i -subgroups are cyclic for $i = 1, 2$. Since each $r_i > 3$, r_i does not divide the order of the Weyl group (which is S_3), and since there are no elements of order $r_1 r_2$, by Lemma 4.2 G contains no subgroup of order $r_1 r_2$ so the result follows from Lemma 3.1. If $q = 3$, we take $|x_1| = 4$ (so of order 2 in the simple group) and $|x_2| = 7$. The maximal subgroups of $\mathrm{SU}_3(3)$ of order divisible by 7 are isomorphic to $\mathrm{SL}_2(7)$, so as above $\langle x_1, x_2 \rangle \simeq \mathrm{SL}_2(7)$.

$G = \mathrm{SU}_4(q)$ with $q = 3^a \geq 3$.

Here $q^4 - 1$ and $q^6 - 1$ have primitive prime divisors r_1, r_2 respectively, each greater than 4, and we argue exactly as for $\mathrm{SU}_3(q)$.

$G = \mathrm{SU}_6(q)$ with $q = 5^a \geq 5$.

Here $q^6 - 1$ and $q^{10} - 1$ have primitive prime divisors r_1, r_2 respectively, each greater than 6, and we argue exactly as for $\mathrm{SU}_3(q)$.

$G = \mathrm{Sp}_4(q)$ with $q = 2^a \geq 2$.

The case $\mathrm{Sp}_4(2)' \cong A_6$ follows from Proposition 3.3, so we assume $q \geq 4$. Let r_2 be a primitive prime divisor of $q^4 - 1$, and note that $r_2 \geq 5$. Then the Sylow r_2 -subgroups are cyclic with normalizers of order $4(q^2 + 1)$. Let $r_1 = 3$. Then any 3-subgroup of G acts reducibly and so cannot be normalized by a subgroup of prime order r_2 . Thus G has no subgroups of order $3r_2^b$ or $3^b r_2$ (with $b > 0$) and Lemma 3.1 applies.

$G = \mathrm{Sp}_6(2)$.

Take $r_1 = 5$ and $r_2 = 7$. Both Sylow subgroups are cyclic and there are no subgroups of order 35, whence the result follows.

$G = \mathrm{Sp}_8(2)$ or $\mathrm{Spin}_8^-(2)$.

Let $r_2 = 17$ and $r_1 = 5$. Then G has no subgroups of order $r_1 r_2^b$ or $r_1^b r_2$ (with $b > 0$) and Lemma 3.1 applies.

$G = \mathrm{Spin}_8^+(q)$ with $q = 2^a$ or 3^a .

Here $q^4 - 1$ has a primitive prime divisor r_1 , and if $q > 2$ then $q^6 - 1$ has a primitive prime divisor r_2 ; if $q = 2$ take $r_2 = 7$. Then each Sylow r_1 -subgroup is abelian and is contained in a maximal torus of order $(q^2 + 1)^2$. It follows that the automizer of any r_1 -subgroup is a 2-group. In particular, the normalizer of any r_1 -subgroup contains no elements of order r_2 . Also, the Sylow r_2 -subgroups are cyclic and, since $r_1 \geq 5$, r_1 does not divide the order of the Weyl group of G , whence by Lemma 4.2 the normalizer of any r_2 -subgroup contains no elements

of order r_1 . Now the result follows from Lemma 3.1.

This completes the proof of the following proposition.

Proposition 4.4. *Theorem B holds for the classical simple groups of Lie type.*

4.2. Finite exceptional groups. We now turn our attention to the exceptional groups. Let $\Phi_m(t)$ denote the m th cyclotomic polynomial.

The general strategy is as follows. In all cases other than ${}^3D_4(q)$, we will choose two maximal tori T_1 and T_2 and suitable primitive prime divisors r_i of $|G|$ such that each T_i is cyclic and contains a Sylow r_i -subgroup of G . Moreover, we choose the r_i so that neither r_i divides the order of the Weyl group (this is not hard to arrange since only the primes 2 and 3 divide the order of the Weyl group unless the prime is 5 and $G = E_n(q)$ or ${}^2E_6(q)$ or the prime is 7 and $G = E_7(q)$ or $E_8(q)$). Then Lemma 3.1 applies to give the result. See the Table for this information (we refer the reader to [GM1, Table 6] and [GM2, Table 1]).

TABLE 1. Cyclic Tori in Exceptional Groups

G	$ T_1 $	$ T_2 $
${}^2B_2(q^2), q^2 \geq 8$	$q^2 + \sqrt{2}q + 1$	$q^2 - \sqrt{2}q + 1$
${}^2G_2(q^2), q^2 \geq 27$	$q^2 + \sqrt{3}q + 1$	$q^2 - \sqrt{3}q + 1$
$G_2(q),$	$q^2 + q + 1$	$q^2 - q + 1$
${}^2F_4(q^2), q^2 \geq 8$	$q^4 + \sqrt{2}q^3 + q^2 + \sqrt{2}q + 1$	$q^4 - \sqrt{2}q^3 + q^2 - \sqrt{2}q + 1$
$F_4(q),$	$q^4 - q^2 + 1$	$\Phi_8(q)$
$E_6(q)$	$\Phi_9(q)$	$\Phi_3(q)\Phi_{12}(q)$
${}^2E_6(q)$	$\Phi_{18}(q)$	$\Phi_6(q)\Phi_{12}(q)$
$E_7(q)$	$\Phi_2(q)\Phi_{18}(q)$	$\Phi_1(q)\Phi_9(q)$
$E_8(q)$	$\Phi_{30}(q)$	$\Phi_{15}(q)$

Finally, consider ${}^3D_4(q)$. Let r_2 be a divisor of $q^4 - q^2 + 1$ and r_1 a divisor of $\Phi_6(q)$ (both of which divide $|{}^3D_4(q)|$). Then there is a cyclic maximal torus T_2 of order $q^4 - q^2 + 1$ containing a Sylow r_2 -group of G . Indeed, T_2 is the centralizer of a Sylow r_2 -subgroup. Similarly, there is a maximal subgroup T_1 of order $\Phi_6(q)(q^2 - 1)$ and this contains a cyclic Sylow r_1 -subgroup. Since $r_2 \geq 13$ and $r_1 \geq 7$, we see there are no subgroups of order r_1r_2 , whence Theorem B holds.

Thus we have proved the following proposition.

Proposition 4.5. *Theorem B holds for the exceptional simple groups of Lie type.*

We are ready now for the proof of Theorem B.

Proof of Theorem B. By the classification of the finite nonabelian simple groups, they belong to one of the following classes: the alternating simple groups, the sporadic simple groups, the classical simple groups of Lie type and the exceptional simple groups of Lie type. Thus Theorem B follows from Proposition 3.4, Proposition 4.4 and Proposition 4.5. \square

5. PROOFS OF THEOREMS A AND C

We will see that Theorem A follows from Theorem B.

Proof of Theorem A. The implications $(1) \Rightarrow (2)$, and $(2) \Rightarrow (3)$ are obvious. We prove that $(3) \Rightarrow (1)$. Thus let G be a finite group such that, if $x, y \in G$ have prime power order, then $\langle x, y^g \rangle$ is solvable for some $g \in G$. We need to prove that G is solvable. Suppose that this is not the case, and let G be a minimal counterexample. Let N be a minimal normal subgroup of G . Note that if $xN \in G/N$ has prime power order, we can replace x by a power of itself and assume that x also has prime power order. Thus, G/N also satisfies hypothesis (3). By the minimality of G it follows that G/N is solvable. Thus, since G is nonsolvable, N is a nonsolvable minimal normal subgroup and so $N = L_1 \times \dots \times L_t \cong L^t$ for some nonabelian simple group L and $t \geq 1$. By Theorem B there exist distinct primes p, q dividing $|L|$ such that $\langle x_1, y_1 \rangle$ is nonsolvable for all $x_1, y_1 \in L_1$ of orders p and q respectively. Let $x = (x_1, \dots, x_t) \in N$ and $y = (y_1, \dots, y_t) \in N$ with each x_i of order p and each y_i of order q . If $g \in G$, then $\langle x, y^g \rangle$ is a subgroup of N for which the projection to each direct factor L_i of N is a subgroup $\langle x_i, y_i^g \rangle$ with $|x_i| = p$, $|y_i^g| = q$, and hence is nonsolvable. In particular, $\langle x, y^g \rangle$ is nonsolvable, a contradiction. \square

Proof of Corollary D. This uses standard reductions for linear groups. Let $G \leq \text{GL}_n(K)$ be a finitely generated linear group for some n and field K . Note that, by a theorem of Lie, Kolchin and Mal'cev [Ro, 15.1.1], a solvable subgroup of $\text{GL}_n(K)$ has derived length at most $f(n)$ for some function f .

Suppose now that, for all $x, y \in G$ there exists $g \in G$ such that $\langle x, y^g \rangle$ is solvable, but that G is not solvable. Then the $(f(n) + 1)$ th term in the derived series for G contains a non-identity element, say w . Let R be a finitely generated subring of K containing all the matrix

entries of the generators for G as well as the inverses of the nonzero entries of $w - I$. Let M be a maximal ideal of R . By the Nullstellensatz, R/M is a finite field. Each element of G has entries in R , and reducing entries modulo M defines a homomorphism φ from G into $\text{GL}_n(R/M)$. By construction, the image $\varphi(w)$ is nontrivial in $\text{GL}_n(R/M)$. By Theorem A, the image $\varphi(G)$ in $\text{GL}_n(R/M)$ is solvable, whence $\varphi(G)$ has derived length at most $f(n)$. This however implies that $\varphi(w)$ is trivial, which is a contradiction. \square

We now prove Corollaries E and F. We give two proofs of Corollary E. Clearly a nilpotent group has the stated property, so we assume that, for each pair p and q of distinct primes, and for all p -elements x and q -elements y in a finite group G , x and y^g commute for some $g \in G$. Moreover we suppose inductively that every group of smaller order with this property is nilpotent.

Proof 1 of Corollary E. It follows that, if $x, y \in G$ have prime power order, then $\langle x, y^g \rangle$ is solvable for some $g \in G$. (This is true if $|x|$ and $|y|$ are powers of the same prime by Sylow's Theorem.) Thus, Theorem A implies that G is solvable. Let N be a minimal normal subgroup of G . Then N is an elementary abelian p -group, for some prime p . By induction, G/N is nilpotent. We claim that N is central in G . Note that this implies that G is nilpotent, as required. Suppose to the contrary that N is not central, and choose $z \in G$ of prime power order with $zC_G(N) \in Z(G/C_G(N))$ of prime order q . Since N is minimal, $C_N(z) = 1$ (since it is normal and properly contained in N). Thus, $q \neq p$ and z is a q -element. Similarly, $C_N(z^g) = 1$ for all $g \in G$ and so, for $x \in N$ of order p , z^g does not commute with x for any $g \in G$. This contradiction proves the claim, and completes the proof. \square

Proof 2 of Corollary E. Let P be a nontrivial Sylow p -subgroup of G . Let $z \in Z(P)$ and set $C = C_G(z)$. If $C = G$ then, by induction, $G/\langle z \rangle$ is nilpotent, and hence also G is nilpotent, as in the first proof. So we may assume that $C \neq G$. Then, by [FKS] applied to the transitive action of G on the cosets of C , there exists $y \in G$ of prime power order that is not conjugate to any element of C . Since $z \in Z(P)$, this means that y is a q -element with $q \neq p$, and this contradicts the hypotheses. \square

Proof of Corollary F. If G is solvable, the result follows from the existence and conjugacy of Hall subgroups. If G is not solvable, then by Theorem A, there exist primes p and q , and a p -element x and a q -element y , such that $\langle x, y^g \rangle$ is not solvable for all $g \in G$. Thus,

$\langle x, y^g \rangle$ has order divisible by at least 3 primes and so it is not a $\{p, q\}$ -group. \square

To prove Theorem C we recall the result of Guralnick-Malle [GM2, Theorem 7.1]:

Theorem 5.1 (Guralnick-Malle). *Let G be a finite almost simple group with socle S . Then there exist conjugacy classes C and D of G such that $C \subset S$ and if $(x, y) \in C \times D$, then $\langle x, y \rangle \geq S$.*

We deduce Theorem C from this result. The proof is similar to the proof of Theorem A but a bit more complicated.

Proof of Theorem C. Let \mathcal{X} be a family of finite groups closed under forming subgroups, quotient groups and extensions. Let G be a minimal counterexample to Theorem C with respect to the family \mathcal{X} . Thus $G \notin \mathcal{X}$ (in particular $G \neq 1$) and, for any conjugacy classes C and D of G , there exists $(x, y) \in C \times D$ such that $\langle x, y \rangle \in \mathcal{X}$.

Note that this means in particular that $G \not\cong Z_p$ for a prime p . Also, since $G \notin \mathcal{X}$ it follows from Theorem 5.1 that G is not a nonabelian simple group. Thus G is not simple. Let N be a minimal normal subgroup of G . Since the hypothesis still holds for G/N , we see that $G/N \in \mathcal{X}$. If N were in \mathcal{X} as well, then since \mathcal{X} is closed under extensions, G would also lie in \mathcal{X} , which is not the case. Hence $N \notin \mathcal{X}$. Suppose next that N is an elementary abelian p -group for some prime p . Then taking C and D to be G -conjugacy classes contained in N , we find that \mathcal{X} contains a nontrivial p -group, and hence contains N , which is a contradiction.

Thus, $N = L_1 \times \dots \times L_t \cong L^t$ for some nonabelian simple group L and $t \geq 1$. Since $N \notin \mathcal{X}$ we note that also $L \notin \mathcal{X}$. Let $H = N_G(L_1)$ and $\bar{H} = N_G(L_1)/C_G(L_1)$. Then \bar{H} is almost simple with socle $\bar{L}_1 \cong L$. By Theorem 5.1, there exist conjugacy classes \bar{C}_1 and \bar{D}_1 of \bar{H} such that $\bar{C}_1 \subset \bar{L}_1$ and every pair in $\bar{C}_1 \times \bar{D}_1$ generates a subgroup containing \bar{L}_1 . Let $\Omega = \{u_1, \dots, u_t\}$ denote a set of left coset representatives for H in G , with $u_1 = 1$. Let C_1 be an H -conjugacy class contained in L_1 which projects to \bar{C}_1 modulo $C_G(L_1)$, and let $C = \{\prod_i^t c_i^{u_i} \mid c_i \in C_1 \text{ for all } i\}$. Then C is a conjugacy class of G and $C \subset N$. Let $d \in H$ be the lift of an element of \bar{D}_1 and set $D = d^G$.

We claim that if $(x, y) \in C \times D$, then $\langle x, y \rangle$ is not in \mathcal{X} . This claim gives a contradiction and completes the proof.

Let $(x, y) \in C \times D$. Conjugating x, y by the same element of G we may assume that $y = d \in H$. Now $x = \prod_i^t c_i^{u_i}$ with each $c_i \in C_1$. Then, in \bar{H} , the image \bar{y} of y is in \bar{D}_1 and the image \bar{x} of x is $\bar{c}_1 \in \bar{C}_1$. Thus,

$\langle \bar{x}, \bar{y} \rangle$ contains \bar{L}_1 and so is not in \mathcal{X} , whence also $\langle x, y \rangle \notin \mathcal{X}$, proving the claim and the theorem. \square

6. FURTHER REMARKS AND EXAMPLES

When considering the two results Theorem B and Theorem 5.1, it is natural to ask whether the classes C and D of Theorem 5.1 can be chosen to consist of elements of prime order; and to ask whether the condition “ $\langle x, y \rangle$ is nonsolvable” in Theorem B can be replaced by “ $\langle x, y \rangle = G$ ”. Examples 6.1 and 6.2 demonstrate that this cannot be done in general. In particular, example 6.2 shows that there are infinitely many finite simple groups of Lie type for which no choice of primes a, b gives the stronger “generation result”.

Example 6.1. Let $G = O^+(8, 2)$. The only primes dividing $|G|$ are 2, 3, 5 and 7. If $r = 2$ or 3, then it is easy to see that if x has order r and the order of y is any prime, then $\langle x, y^g \rangle$ is solvable for some g . Also, if $r_1 = 5$ and $r_2 = 7$, then there exist x_i of order r_i such that $\langle x_1, x_2 \rangle = 2^6.A_8$.

Example 6.2. Let $G = \text{Sp}_{4m}(q)$, with $q = 2^f \geq 2$, and let $H = O_{4m}^-(q) < G$. Then any prime dividing $|G|$ also divides $|H|$. Thus, for any pair of prime divisors of $|G|$, there are elements x, y of these prime orders such that $\langle x, y \rangle \leq H$.

Note that $A_5 < A_6$ is another example ($S_6 \cong \text{Sp}_4(2)$ so this actually fits into the previous family). Similarly, if n is not prime, then A_{n-1} and A_n have precisely the same prime divisors.

We make the following conjecture.

Conjecture. *With finitely many exceptions, if G is a finite simple group of Lie type, then there exist conjugacy classes C and D of G consisting of elements of prime order such that $\langle x, y \rangle = G$ for any $(x, y) \in C \times D$.*

Note that the infinite family of groups in Example 6.2 does not contradict the conjecture. Namely, let r_2 be a primitive prime divisor of $q^{4m} - 1$ and let r_1 be a primitive prime divisor of $q^{2m} - 1$. Let x_1 be an element of order r_1 with trivial fixed space on the natural G -module. Then x_1 is not conjugate to an element of H . Let x_2 be an element of order r_2 . It is easy to see that generically, $\langle x_1, x_2^g \rangle = G$ for all $g \in G$.

The conjecture does fail for A_n (the smallest counterexample with $n > 6$ is $n = 210$), but it does seem likely to hold for a density 1 subset (one can prove that it is true for a subset of positive density). We do not know an example of an alternating group which cannot be

generated by each pair in $C \times D$, for some conjugacy classes C and D consisting of elements of prime power order.

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